

# CONSTRUCTION OF CUSP FORMS USING RANKIN-COHEN BRACKETS

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**ABSTRACT.** For a fix modular form  $g$  and a non negative ineteger  $\nu$ , by using Rankin-Cohen bracket we first define a linear map  $T_{g,\nu}$  on the space of modular forms. We explicitly compute the adjoint of this map and show that the  $n$ -th Fourier coefficients of the image of the cusp form  $f$  under this map is, upto a constant a special value of Rankin-Selberg convolution of  $f$  and  $g$ . This is a generalization of the work due to W. Kohnen (Math. Z., 207, (1991), 657-660) and S. D. Herrero (Ramanujan J., 36(2014), no.3, 529-536) in the case of integral weight modular forms to half integral weight modular forms. As a consequence we get non-vanishing of special value of certain Rankin- Selberg convolution of modular forms.

## 1. INTRODUCTION

W. Kohnen [7] constructed cusp forms whose Fourier coefficients are given by special values of certain Dirichlet series by computing the adjoint of the product map by a fixed cusp form with respect to the usual Petersson scalar product. This result has been generalized by several authors to other automorphic forms (see the list [1, 8, 9, 10, 12]). The work of Kohnen has been generalized by S. D. Herrero [3], where the author constructed the cusp forms by computing the adjoint of the map constructed using the Rankin-Cohen brackets by a fixed cusp form instead of the product map. Recently, the work of S. D. Herrero [3] has been generalised by first author and B. Sahu to the case of Jacobi forms [5] which also generalises the result of H. Sakata [10]. In this article we extend the work of S. D. Herrero to the case of half integral weight modular forms. We apply this result to get non-vanishing of special value of certain Rankin- Selberg convolution of modular forms.

## 2. PRELIMINARIES

**2.1. Elliptic Modular Forms.** Let  $\mathcal{H}$  be the complex upper half-plane and  $\Gamma$  be a congruence subgroup of the full modular group  $SL_2(\mathbb{Z})$ . For  $k \in \mathbb{Z}$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , define the slash operator as follows;

$$f |_k \gamma(z) := (cz + d)^{-k} f(\gamma z), \text{ where } \gamma z = \frac{az + b}{cz + d}.$$

Let  $M_k(\Gamma, \chi)$  (respectively  $S_k(\Gamma, \chi)$ ) denote the space of modular forms (resp. cusp forms) of integral weight  $k$  and character  $\chi$  for  $\Gamma$ , i.e., for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ,  $f |_k \gamma(z) =$

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$\chi(d)f(z)$ , and holomorphic at cusps of  $\Gamma$ .

We define the Petersson scalar product on  $S_k(\Gamma, \chi)$  as follows;

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} (Im(z))^k d^*z,$$

where  $z = x + iy$  and  $d^*z = \frac{dx dy}{y^2}$  is an invariant measure under the action on  $\Gamma$  on  $\mathcal{H}$ . For more details on the theory of modular forms, we refer to [6].

## 2.2. Poincaré series.

**Definition 2.1.** *Let  $n$  be a positive integer. The  $n$ -th Poincaré series of integer weight  $k$  is defined by*

$$P_{k,n}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z} |_k \gamma, \quad (1)$$

where  $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z} \right\}$ . It is well known that  $P_{k,n} \in S_k(\Gamma)$  for  $k > 2$ .

This series has the following property.

**Lemma 2.2.** *Let  $f \in S_k(\Gamma)$  with Fourier expansion  $f(z) = \sum_{m=1}^{\infty} a(m)q^m$ . Then*

$$\langle f, P_{k,n} \rangle = \alpha_{k,n} a(n), \quad \text{where } \alpha_{k,n} = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}}. \quad (2)$$

**2.3. Modular Forms of Half Integral Weight.** Let  $\Gamma = \Gamma_0(4)$ . For  $k \in \mathbb{Z}$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , define the slash operator as follows;

$$f|_{k+\frac{1}{2}} \gamma(z) := \left( \frac{c}{d} \right) \left( \frac{-4}{d} \right)^{k+\frac{1}{2}} (cz + d)^{-k-\frac{1}{2}} f(\gamma z),$$

where  $\left( \frac{c}{d} \right)$  is the Kronecker symbol.

Let  $M_{k+\frac{1}{2}}(\Gamma, \chi)$  (resp.  $S_{k+\frac{1}{2}}(\Gamma, \chi)$ ) denote the space of modular forms (resp. cusp forms) of weight  $k + \frac{1}{2}$  and character  $\chi$  for  $\Gamma$ , i.e., for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ,  $f|_{k+\frac{1}{2}} \gamma(z) = \chi(d)f(z)$ , and holomorphic (resp. vanish) at cusps of  $\Gamma$ .

We define the Petersson scalar product on  $S_{k+\frac{1}{2}}(\Gamma, \chi)$  as follows;

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} (Im(z))^{k+\frac{1}{2}} d^*z,$$

where  $z = x + iy$ . The spaces  $S_k(\Gamma, \chi)$  and  $S_{k+\frac{1}{2}}(\Gamma, \chi)$  are finite dimensional Hilbert spaces. For more details on the theory of modular forms of half integral weight, we refer to [6] and [11].

#### 2.4. Poincaré series of half integral weight.

**Definition 2.3.** Let  $n$  be a positive integer. The  $n$ -th Poincaré series of weight  $k + \frac{1}{2}$ , where  $k \in \mathbb{Z}$  is defined by

$$P_{k+\frac{1}{2},n}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z} \Big|_{k+\frac{1}{2}} \gamma. \quad (3)$$

It is well known that  $P_{k+\frac{1}{2},n} \in S_{k+\frac{1}{2}}(\Gamma)$  for  $k > 2$ .

This series has the following property.

**Lemma 2.4.** Let  $f \in S_{k+\frac{1}{2}}(\Gamma)$  with Fourier expansion  $f(z) = \sum_{m=1}^{\infty} a(m)q^m$ . Then

$$\langle f, P_{k+\frac{1}{2},n} \rangle = \tilde{\alpha}_{k,n} a(n), \quad \text{where } \tilde{\alpha}_{k,n} = \frac{\Gamma(k - \frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}}. \quad (4)$$

The following lemmas tell about the growth of the Fourier coefficients of a modular form.

**Lemma 2.5.** [4] If  $f \in M_k(\Gamma, \chi)$  with Fourier coefficients  $a(n)$ , then

$$a(n) \ll |n|^{k-1+\epsilon},$$

and moreover, if  $f$  is a cusp form, then

$$a(n) \ll |n|^{\frac{k}{2}-\frac{1}{4}+\epsilon}.$$

**Lemma 2.6.** If  $f \in M_{k+\frac{1}{2}}(\Gamma, \chi)$  with Fourier coefficients  $a(n)$ , then

$$a(n) \ll |n|^{k-\frac{1}{2}+\epsilon},$$

and moreover, if  $f \in S_{k+\frac{1}{2}}(\Gamma, \chi)$  is a cusp form, then

$$a(n) \ll |n|^{\frac{k}{2}+\epsilon}.$$

**2.5. Rankin-Cohen Brackets.** Let  $k$  and  $l$  be real numbers and  $\nu \geq 0$  be an integer. Let  $f$  and  $g$  be two complex valued holomorphic functions on  $\mathcal{H}$ . Define the  $\nu$ -th Rankin-Cohen bracket of  $f$  and  $g$  by

$$[f, g]_\nu := \sum_{r=0}^{\nu} C_r(k, l; \nu) D^r f D^{\nu-r} g, \quad (5)$$

where  $D^r f = \frac{1}{(2\pi i)^r} \frac{d^r f}{dz^r}$  and  $C_r(k, l; \nu) = (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)}$  and  $\Gamma(x)$  is the usual Gamma function.

*Remark 2.1.* It is easy to verify that

$$[f|_k \gamma, g|_l \gamma]_\nu = [f, g]|_{k+l+2\nu} \gamma, \quad \forall \gamma \in \Gamma. \quad (6)$$

*Remark 2.2.* We note that the 0-th Rankin-Cohen bracket is the usual product of modular forms i.e.,  $[f, g]_0 = fg$ .

**Theorem 2.7.** [2] *Let  $\nu \geq 0$  be an integer and  $f \in M_k(\Gamma, \chi_1)$  and  $g \in M_l(\Gamma, \chi_2)$ . Then  $[f, g]_\nu \in M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$ ,*

$$\text{where } \chi = \begin{cases} 1, & \text{if both } k, l \in \mathbb{Z}, \\ \chi_{-4}^k, & \text{if } k \in \mathbb{Z} \text{ and } l \in \mathbb{Z} + \frac{1}{2}, \\ \chi_{-4}^l, & \text{if } k \in \mathbb{Z} + \frac{1}{2} \text{ and } l \in \mathbb{Z}, \\ \chi = \chi_{-4}^{k+l} & \text{if both } k, l \in \mathbb{Z} + \frac{1}{2}, \end{cases}$$

Moreover if  $\nu > 0$ , then  $[f, g]_\nu \in S_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$ . In fact,  $[\ , \ ]_\nu$  is a bilinear map from  $M_k(\Gamma, \chi_1) \times M_l(\Gamma, \chi_2)$  to  $M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$ . Here  $\chi_{-4}$  is the character defined by  $\chi_{-4}(x) = (\frac{-4}{x})$ .

Let  $k, l \in \frac{\mathbb{Z}}{2}$  and  $\nu \geq 0$  be integers and  $\Gamma$  be a congruence subgroup of the full modular group  $SL_2(\mathbb{Z})$ . Also assume that  $\Gamma \subseteq \Gamma_0(4)$  if either of  $k$  or  $l$  is non integer. For a fixed  $g \in M_l(\Gamma, \chi_2)$ , we define the map

$$T_{g,\nu} : S_k(\Gamma) \rightarrow S_{k+l+2\nu}(\Gamma, \chi_2)$$

defined by  $T_{g,\nu}(f) = [f, g]_\nu$ .  $T_{g,\nu}$  is a  $\mathbb{C}$ -linear map of finite dimensional Hilbert spaces and therefore has an adjoint map  $T_{g,\nu}^* : S_{k+l+2\nu}(\Gamma, \chi_2) \rightarrow S_k(\Gamma)$  such that

$$\langle f, T_{g,\nu}(h) \rangle = \langle T_{g,\nu}^*(f), h \rangle, \quad \forall f \in S_{k+l+2\nu}(\Gamma, \chi_2) \text{ and } h \in S_k(\Gamma).$$

In [3] S.D. Herrero computed the adjoint map for the case when  $k, l \in \mathbb{Z}, \Gamma = SL_2(\mathbb{Z})$  and  $\chi_2$  is the trivial character.

**Theorem 2.8.** [3] *Let  $k \geq 6$  and  $l$  be natural numbers and  $\nu \geq 0$ . Let  $g \in M_l(SL_2(\mathbb{Z}))$  with Fourier expansion*

$$g(z) = \sum_{m=0}^{\infty} b(m)q^m.$$

*Suppose that either (a)  $g$  is a cusp form or (b)  $g$  is not cusp form and  $l < k - 3$ . Then the image of any cusp form  $f \in S_{k+l+2\nu}(SL_2(\mathbb{Z}))$  with Fourier expansion*

$$f(z) = \sum_{m=1}^{\infty} a(m)q^m$$

*under  $T_{g,\nu}^*$  is given by*

$$T_{g,\nu}^*(f)(z) = \sum_{n=1}^{\infty} c(n)q^n,$$

*where*

$$c(n) = \beta(k, l, \nu; n) L_{f,g,\nu,n}(\gamma), \tag{7}$$

*where  $L_{f,g,\nu,n}$  is the  $L$ -function associated with  $f$  and  $g$ , defined by for  $s \in \mathbb{C}$ ,*

$$L_{f,g,\nu,n}(s) = \sum_{m=1}^{\infty} \frac{a(n+m) \overline{b(m)} \alpha(k, l, \nu, n, m)}{(n+m)^s}$$

with

$$\alpha(k, l, \nu, n, m) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)} n^r m^{\nu-r}$$

and

$$\gamma = k + l + 2\nu - 1, \beta(k, l, \nu; n) = \frac{\Gamma(k+l+2\nu-1) n^{k-1}}{\Gamma(k-1)(4\pi)^{l+2\nu}}.$$

*Remark 2.3.* One can prove the similar result for the case when  $\Gamma$  is a congruence subgroup of level  $N$  and  $\chi_2$  is any character mod  $N$  using the technique used in proof of Theorem 3.1.

### 3. STATEMENT OF THE THEOREM

Consider the following maps:

- (1)  $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \rightarrow S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ , with  $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ ,
- (2)  $T_{g,\nu} : S_k(\Gamma) \rightarrow S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$ , with  $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ ,
- (3)  $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \rightarrow S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$ , with  $g \in M_l(\Gamma, \chi_2)$

We exhibit explicitly the Fourier coefficients of  $T_{g,\nu}^*(f)$  for  $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$  in (1) and by using the same method, we can find the analogous maps in (2) and (3) (see the remark 3.1). These involve special values of certain Dirichlet series of Rankin- Selberg type associated to  $f$  and  $g$ . We now state the main theorem.

**Theorem 3.1.** *Let  $k$  and  $l$  be natural numbers and  $\nu \geq 0$ . Let  $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$  with Fourier expansion*

$$g(z) = \sum_{m=0}^{\infty} b(m)q^m.$$

*Suppose that either (a)  $g$  is a cusp form and  $k > 2$  or (b)  $g$  is not cusp form and  $l < k - \frac{3}{2}$ . Then the image of any cusp form  $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$  with Fourier expansion*

$$f(z) = \sum_{m=1}^{\infty} a(m)q^m$$

*under  $T_{g,\nu}^*$  is given by*

$$T_{g,\nu}^*(f)(z) = \sum_{n=1}^{\infty} c(n)q^n,$$

*where*

$$c(n) = \beta(k, l, \nu; n) L_{f,g,\nu,n}(\gamma), \tag{8}$$

*where*

$$\gamma = k + l + 2\nu, \beta(k, l, \nu; n) = \frac{\Gamma(k+l+2\nu) n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})(4\pi)^{l+2\nu+\frac{1}{2}}}$$

*and  $L_{f,g,\nu,n}(\gamma)$  is defined in Theorem 2.8.*

*Remark 3.1.* We have the similar results for the map in (2) with

$$\gamma = k + l + 2\nu - \frac{1}{2}, \text{ and } \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu - \frac{1}{2}) n^{k-1}}{\Gamma(k-1) (4\pi)^{l+2\nu+\frac{1}{2}}},$$

and for the map in (3) with

$$\gamma = k + l + 2\nu - \frac{1}{2}, \text{ and } \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu - \frac{1}{2}) n^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2}) (4\pi)^{l+2\nu}},$$

with the assumption that either (a)  $g$  is a cusp form and  $k > 3$  or (b)  $g$  is not cusp form and  $l < k - 2$ .

*Remark 3.2.* Using Lemma 2.5 and Lemma 2.6 one can show that the series appearing in (8) converges.

#### 4. PROOF OF THEOREM 3.1

We need the following lemma to proof the main theorem.

**Lemma 4.1.** *Using the same notation in Theorem 3.1, we have*

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash \mathcal{H}} |f(z) \overline{[e^{2\pi i n z} |_k \gamma, g]_\nu} (Im(z))^{k+l+2\nu+1} | d^* z$$

converges.

*Proof.* The proof is similar to Lemma 1 in [3]. □

Now we give a proof of Theorem 3.1. Put

$$T_{g,\nu}^*(f)(z) = \sum_{n=1}^{\infty} c(n) q^n.$$

Now, we consider the  $n$ -th Poincaré series of weight  $k + \frac{1}{2}$  as given in (3). Then using the Lemma 2.4, we have

$$\langle T_{g,\nu}^* f, P_{k+\frac{1}{2},n} \rangle = \tilde{\alpha}_{k,n} c(n),$$

where

$$\tilde{\alpha}_{k,n} = \frac{\Gamma(k - \frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}}.$$

On the other hand, by definition of the adjoint map we have

$$\langle T_{g,\nu}^* f, P_{k+\frac{1}{2},n} \rangle = \langle f, T_{g,\nu}(P_{k+\frac{1}{2},n}) \rangle = \langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle.$$

Hence we get

$$c(n) = \frac{(4\pi n)^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2})} \langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle. \quad (9)$$

By definition,

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{[P_{k+\frac{1}{2},n}, g]_\nu(z)} (Im(z))^{k+l+2\nu+1} d^*z \\
&= \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{\left[ \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z} \tilde{[}_{k+\frac{1}{2}} \gamma, g]_\nu(z) \right]} (Im(z))^{k+l+2\nu+1} d^*z \\
&= \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(z) \overline{[e^{2\pi i n z} \tilde{[}_{k+\frac{1}{2}} \gamma, g]_\nu(z)} (Im(z))^{k+l+2\nu+1} d^*z.
\end{aligned}$$

By Lemma 4.1, we can interchange the sum and integration in  $\langle f, [P_{k,n}, g]_\nu \rangle$ . Hence we get,

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{[e^{2\pi i n z} \tilde{[}_{k+\frac{1}{2}} \gamma, g]_\nu(z)} (Im(z))^{k+l+2\nu+1} d^*z.$$

Since  $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ ,  $\tilde{g}|_{l+\frac{1}{2}} \gamma = \chi_2(d)g(z)$ , for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ . Therefore

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{[e^{2\pi i n z} \tilde{[}_{k+\frac{1}{2}} \gamma, \frac{1}{\chi_2(d)} \tilde{g}|_{l+\frac{1}{2}} \gamma]_\nu(z)} (Im(z))^{k+l+2\nu+1} d^*z \\
&= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} \left( \frac{(-\frac{4}{d})^{k+l+1}}{\chi_2(d)} \right) \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{[e^{2\pi i n z} |_{k+\frac{1}{2}} \gamma, g|_{l+\frac{1}{2}} \gamma]_\nu(z)} (Im(z))^{k+l+2\nu+1} d^*z.
\end{aligned}$$

Using the change of variable  $z$  to  $\gamma^{-1}z$  in each integral,  $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$  equals

$$\sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} \left( \frac{(-\frac{4}{d})^{k+l+1}}{\chi_2(d)} \right) \int_{\Gamma \backslash \mathcal{H}} f(\gamma^{-1}z) \overline{[e^{2\pi i n z} |_{k+\frac{1}{2}} \gamma, g|_{l+\frac{1}{2}} \gamma]_\nu(\gamma^{-1}z)} (Im(\gamma^{-1}z))^{k+l+2\nu+1} d^*(\gamma^{-1}z).$$

Since  $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ ,  $f(\gamma^{-1}z) = \chi_2(d)\chi(d)(cz+d)^{k+l+2\nu+1}f(z)$ , for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , and hence

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} \left( \frac{(-\frac{4}{d})^{k+l+1}}{\chi_2(d)} \right) \int_{\Gamma \backslash \mathcal{H}} \chi_2(a)\chi(a)(-cz+a)^{k+l+2\nu+1} f(z) \\
&\times \overline{(-cz+a)^{k+l+2\nu+1}([e^{2\pi i n z} |_{k+\frac{1}{2}} \gamma, g|_{l+\frac{1}{2}} \gamma]_\nu |_{k+l+2\nu+1} \gamma^{-1})(z)} \left( \frac{Im(z)}{|-cz+a|^2} \right)^{k+l+2\nu+1} d^*z.
\end{aligned}$$

Now using (6), we get

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma} \overline{\left( \frac{(-4)}{d} \right)^{k+l+1}} \chi_2(a) \chi(a) \int_{\gamma \Gamma \setminus \mathcal{H}} f(z) \overline{[e^{2\pi i n z}, g]_\nu} (Im(z))^{k+l+2\nu+1} d^* z \\
&= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma} \overline{\left( \frac{(-4)}{d} \right)^{k+l+1}} \chi_2(a) \left( \frac{-4}{a} \right)^{k+l+1} \int_{\gamma \Gamma \setminus \mathcal{H}} f(z) \overline{[e^{2\pi i n z}, g]_\nu} (Im(z))^{k+l+2\nu+1} d^* z.
\end{aligned}$$

The quantity appearing before integral is equals to 1, for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma$ , hence we get

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \int_{\gamma \Gamma \setminus \mathcal{H}} f(z) \overline{[e^{2\pi i n z}, g]_\nu} (Im(z))^{k+l+2\nu+1} d^* z.$$

Now using Rankin unfolding argument, we have

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \int_{\Gamma_\infty \setminus \mathcal{H}} f(z) \overline{[e^{2\pi i n z}, g]_\nu} (Im(z))^{k+l+2\nu+1} d^* z \\
&= \int_{\Gamma_\infty \setminus \mathcal{H}} f(z) \sum_{r=0}^{\nu} C_r(k, l; \nu) \overline{D^r(e^{2\pi i n z}) D^{\nu-r}(g)} (Im(z))^{k+l+2\nu+1} d^* z
\end{aligned} \tag{10}$$

Now replacing  $f$  and  $g$  by their Fourier series in (10),  $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$  equals

$$\begin{aligned}
&\sum_{r=0}^{\nu} C_r(k, l; \nu) \int_{\Gamma_\infty \setminus \mathcal{H}} \left( \sum_s a(s) e^{2\pi i s z} \right) n^r \overline{e^{2\pi i n z}} m^{\nu-r} \overline{b(m)} \overline{e^{2\pi i m z}} (Im(z))^{k+l+2\nu+1} d^* z \\
&= \int_{\Gamma_\infty \setminus \mathcal{H}} \sum_s \sum_m \alpha(k, l, \nu, n, m) a(s) \overline{b(m)} e^{2\pi i s z} \overline{e^{2\pi i n z}} \overline{e^{2\pi i m z}} (Im(z))^{k+l+2\nu+1} d^* z \\
&= \sum_s \sum_m \alpha(k, l, \nu, n, m) a(s) \overline{b(m)} \int_{\Gamma_\infty \setminus \mathcal{H}} e^{2\pi i s z} \overline{e^{2\pi i n z}} \overline{e^{2\pi i m z}} (Im(z))^{k+l+2\nu+1} d^* z.
\end{aligned}$$

A fundamental domain for the action of  $\Gamma_\infty$  on  $\mathbb{H}$  is given by  $[0, 1] \times [0, \infty)$ . Integrating on this region after substituting  $z = x + iy$ ,

$$\begin{aligned}
\langle f, [P_{k,n}, g]_\nu \rangle &= \sum_s \sum_m \alpha(k, l, \nu, n, m) a(s) \overline{b(m)} \int_0^1 \int_0^\infty e^{2\pi i (s-n-m)x} e^{-2\pi(\alpha+n+m)y} y^{k+l+2\nu-1} dx dy \\
&= \sum_m \alpha(k, l, \nu, n, m) a(n+m) \overline{b(m)} \int_0^\infty e^{-4\pi(n+m)y} y^{k+l+2\nu-1} dy \\
&= \frac{\Gamma(k+l+2\nu)}{(4\pi)^{k+l+2\nu}} \sum_m \frac{a(n+m) \overline{b(m)} \alpha(k, l, \nu, n, m)}{(n+m)^{k+l+2\nu}}.
\end{aligned}$$



Now substituting the above value of  $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$  in (9), we get the required expression for  $c(n)$  given in Theorem 3.1.

## 5. APPLICATIONS

Consider the linear map  $T_{g,\nu}^* \circ T_{g,\nu}$  on  $S_k(\Gamma)$  with  $g(z) \in M_l(\Gamma, \chi_2)$ . If  $\lambda$  is a eigenvalue of  $T_{g,\nu}^* \circ T_{g,\nu}$ , then  $\lambda \geq 0$ . Suppose that  $S_k(\Gamma)$  is one dimensional space generated by  $f(z) = \sum_m a(n)q^n$ . Then  $T_{g,\nu}^* \circ T_{g,\nu}(h) = \lambda f, \forall h \in S_k(\Gamma)$ . In particular,  $T_{g,\nu}^* \circ T_{g,\nu}(f) = \lambda f$  with  $\lambda \geq 0$  and if we write  $T_{g,\nu}^* \circ T_{g,\nu}(f) = \sum_n c(n)q^n$  then

$$c(n) = \frac{\Gamma(k+l+2\nu-1)}{\Gamma(k-1)} \frac{n^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(n+m)\overline{b(m)} \alpha(k, l, \nu, n, m)}{(n+m)^{k+l+2\nu-1}},$$

where  $a_{T_{g,\nu}(f)}(n)$  is the  $n$ -th Fourier coefficient of  $T_{g,\nu}(f) = [f, g]_\nu$ . If  $a(m_0)$  is the first non-zero Fourier coefficient of  $f$  then by comparing the Fourier coefficients in  $T_{g,\nu}^* \circ T_{g,\nu}(f) = \lambda f$ , we have

$$\lambda = \frac{\Gamma(k+l+2\nu-1)}{a(m_0)\Gamma(k-1)} \frac{m_0^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(m_0+m)\overline{b(m)} \alpha(k, l, \nu, m_0, m)}{(m_0+m)^{k+l+2\nu-1}} \geq 0.$$

In particular, if we take  $l = 0, k = 6$  and  $\nu = 0$  with  $g(z) = \theta(z) = \sum_n q^{n^2}$  and the unique newform  $\Delta_{4,6}(z) = \sum_n \tau_{4,6}(n)q^n \in S_6(\Gamma_0(4))$ , in case (2) then  $m_0 = 1, \alpha(k, l, \nu, m_0, m) = 1$ , and

$$\lambda = \frac{\Gamma(\frac{11}{2})}{\Gamma(5)2\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

or

$$\sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0. \quad (11)$$

Now  $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$  is the  $(m+1)$ -th Fourier coefficient of  $\theta(z)\Delta_{4,6}(z)$  and equals to  $\sum_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)$ . Putting the value of  $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$  in (11), we have

$$\sum_{m=1}^{\infty} \frac{\left( \sum_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r) \right) \overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

or

$$\sum_{m=1}^{\infty} \frac{\left( \sum_{r=1}^{m^2+1} \tau_{4,6}(m^2+1-r^2) \right)}{(m^2+1)^{\frac{11}{2}}} > 0.$$

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